

Standard model cross-over on the latticeMichela D'Onofrio^{1,2} and Kari Rummukainen²¹*Institute for Theoretical Physics, Albert Einstein Center, University of Bern,
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With the physical Higgs mass the standard model symmetry restoration phase transition is a smooth cross-over. We study the thermodynamics of the cross-over using numerical lattice Monte Carlo simulations of an effective $SU(2) \times U(1)$ gauge + Higgs theory, significantly improving on previously published results. We measure the Higgs field expectation value, thermodynamic quantities like pressure, energy density, speed of sound and heat capacity, and screening masses associated with the Higgs and Z fields. While the cross-over is smooth, it is very well defined with a width of only ~ 5 GeV. We measure the cross-over temperature from the maximum of the susceptibility of the Higgs condensate, with the result $T_c = 159.5 \pm 1.5$ GeV. Outside of the narrow cross-over region the perturbative results agree well with nonperturbative ones.

DOI: [10.1103/PhysRevD.93.025003](https://doi.org/10.1103/PhysRevD.93.025003)**I. INTRODUCTION**

The LHC particle accelerator has spectacularly confirmed the standard model of the particle physics: the current combined results from ATLAS and CMS experiments point toward a Higgs boson with a mass 125.1 ± 0.3 GeV [1], and no direct experimental evidence of beyond the Standard Model physics has been observed.

If the electroweak scale physics is fully included in the standard model, we are in the position to have a complete description of the high-temperature electroweak phase transition between the low-temperature broken phase, where the Higgs field has a nonvanishing expectation value, and the high-temperature symmetric phase.¹ The overall nature of the transition was settled already in 1995–98 using lattice simulations [2–7], which indicate a first-order phase transition for Higgs masses $\lesssim 72$ GeV, and a smooth cross-over otherwise. Most of these simulations were done using an effective 3-dimensional theory of the full standard model, derived using perturbation theory. The physics of the transition is nonperturbative due to the infrared singularities arising at momentum scales $\lesssim g^2 T$ [8,9]. These modes are fully captured in effective 3-dimensional theories [10–17], which provide an economical and accurate way to study the nonperturbative physics at the cross-over.

A cross-over means that the early Universe evolved smoothly from symmetric to broken phase without deviating significantly from the thermal equilibrium, rendering e.g. electroweak baryogenesis [18,19] ineffective.

Nevertheless, the cross-over could still influence other processes going on at the same time. An important example is the rate of $B + L$ violation, sphaleron rate, which switches off somewhat below the cross-over temperature [20,21] and may affect the baryon number generation in leptogenesis [22,23]. Decoupling of the dark matter may also be affected by the details of the standard model equation of state in certain scenarios [24,25].

The standard model cross-over with the physical Higgs mass of 125 GeV was recently studied on the lattice in Ref. [20]. The cross-over temperature was reported to be $T_c = 159.5 \pm 1.5$ GeV. However, the focus of this study was the sphaleron rate as a function of the temperature, and for quantities related to thermodynamics the results were limited: as an example, the Higgs field expectation value $\langle \phi^\dagger \phi \rangle$ was determined only at one lattice spacing, and the effects of the hypercharge $U(1)$ field were neglected.

Recently, Laine and Meyer [26] used up to 3-loop perturbative computations and existing results [27,28] to derive a relation between the trace anomaly of the standard model and the Higgs field expectation value in the 3-dimensional effective theory. Using the numerical results from Ref. [20], they calculated several thermodynamic quantities across the cross-over. The combination of perturbative calculations and lattice simulations avoids the infrared problems which make purely perturbative analysis unreliable near the cross-over. The setup is related to the computation of the QCD pressure using effective theory [29]; however, for the standard model electroweak sector the couplings are smaller and the method can be expected to have much better accuracy.

In this work our goal is to improve on the state of the art in lattice simulations of the standard model cross-over: we include the $U(1)$ gauge field, and we use several lattice

¹Although there is no real symmetry breaking phase transition, we use the conventional labels “broken” and “symmetric” to refer to the low- and high-temperature phases, respectively.

spacings and large volumes, enabling us to do reliable continuum extrapolation. The model we simulate is an effective three-dimensional theory containing $SU(2) \times U(1)$ gauge fields and a Higgs scalar. This particular model has been used previously to study the standard model phase transition [30], also with an external hypermagnetic field [31]. However, these older studies used unphysical Higgs masses.

As discussed above, measuring the Higgs condensate is essential for obtaining nonperturbative contributions to thermodynamic quantities. Thus, we pay special attention to precise measurement of the gauge invariant expectation value $\langle \phi^\dagger \phi \rangle$ and its susceptibility. The maximum of the susceptibility allows us to determine the pseudocritical temperature accurately, at $T_c = 159.6 \pm 0.1 \pm 1.5$ GeV. The first error is due to the precision of the lattice simulations and the second is a conservative estimate of the uncertainty of the effective theory description [16]. The pseudocritical temperature is completely consistent with the result in Ref. [20].

We use the condensate and the susceptibility to obtain several thermodynamic quantities across the cross-over: energy density, pressure, heat capacity, speed of sound, and equation of state parameter. For most quantities the magnitude of the effects of the cross-over are small, only at a percent level, but nevertheless clearly visible. We also determine the Higgs and W^3 screening masses and the $\gamma - Z$ mixing. The emerging picture is fully consistent with a smooth and regular cross-over. The cross-over region, where observables deviate significantly from low- or high-temperature behavior, is remarkably narrow, between 157 and 162 GeV.

This paper is organized as follows. In Sec. II we describe the effective theory and in Sec. III its implementation on the lattice. The Higgs condensate and its susceptibility are discussed in Sec. IV, fundamental thermodynamic observables in Sec. V and the screening masses in Sec. VI. We conclude in Sec. VII.

II. EFFECTIVE THREE-DIMENSIONAL DESCRIPTION

At temperatures of order 100 GeV the gauge couplings in the standard model are small, and the Euclidean path integral contains a parametric hierarchy of energy scales: πT , gT and $g^2 T$. While the harder scales can be reliably treated with perturbation theory, at $k \sim g^2 T$ we have to face the nonperturbative infrared physics [8]. The nonperturbative physics can be captured into an effective theory for the soft $k \sim g^2 T$ scales, obtained by integrating over the harder scales using well-defined perturbative methods. This effective theory is purely bosonic and three-dimensional. Concretely, the actual “integration” is done by writing down a general 3-dimensional superrenormalizable effective theory and matching perturbatively computed

two-, three-, and four-point functions in the effective theory and in the original theory, and so fixing the parameters of the effective theory. The detailed description of the derivation of the theory can be found in Refs. [15,16].

The Lagrangian of the effective theory is the 3-dimensional $SU(2) \times U(1)$ gauge theory with a Higgs field

$$L = \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{4} B_{ij} B_{ij} + (D_i \phi)^\dagger D_i \phi + m_3^2 \phi^\dagger \phi + \lambda_3 (\phi^\dagger \phi)^2, \quad (1)$$

where

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i - g_3 [A_i, A_j], & A_i &= \frac{1}{2} \sigma_a A_i^a \\ B_{ij} &= \partial_i B_j - \partial_j B_i \\ D_i &= \partial_i + ig_3 A_i + ig_3' B_i/2. \end{aligned} \quad (2)$$

Here A_i and B_i are the 3-dimensional $SU(2)$ and $U(1)$ gauge fields, g_3^2 and $g_3'^2$ the dimensionful $SU(2)$ and $U(1)$ couplings, and ϕ a complex doublet. The $SU(2) \times U(1)$ local gauge transformation is

$$\phi(x) \rightarrow e^{i\alpha(x)} G(x) \phi(x). \quad (3)$$

The parameters appearing in the effective theory are dimensionful. If we take one of the parameters, say g_3^2 , to set the scale, the dynamics then depends on the three dimensionless parameters x , y and z , defined as

$$x \equiv \frac{\lambda_3}{g_3^2}, \quad y \equiv \frac{m_3^2}{g_3^4}, \quad z \equiv \frac{g_3'^2}{g_3^2}. \quad (4)$$

The four parameters g_3^2 , $g_3'^2$, λ_3 and m_3^2 are definite perturbatively computable functions of the standard model parameters ($\alpha_s(M_W)$, G_F , M_{Higgs} , M_W , M_Z , M_{top} , Weinberg angle), and the temperature T , and are shown in Fig. 1 as functions of the temperature over the relevant temperature range. Depending on the quantity studied, the accuracy of the effective theory can be estimated to be at $\sim 1\%$ level, as discussed in Refs. [16,32].

The overall energy scale is set by the dimensionful $SU(2)$ coupling constant g_3^2 . As can be seen from Fig. 1, it has the value $0.393T$ to better than 0.2% accuracy over the relevant temperature range. However, in our analysis we keep its full T -dependent form, which is tabulated in [33].

From Fig. 1 we see that only y has large temperature dependence. Indeed, from the effective theory point of view it is natural to choose y as the temperature variable, although we present our results in terms of the physical temperature. The transition is expected to happen near $y = 0$, which occurs at $T = 162.1$ GeV.

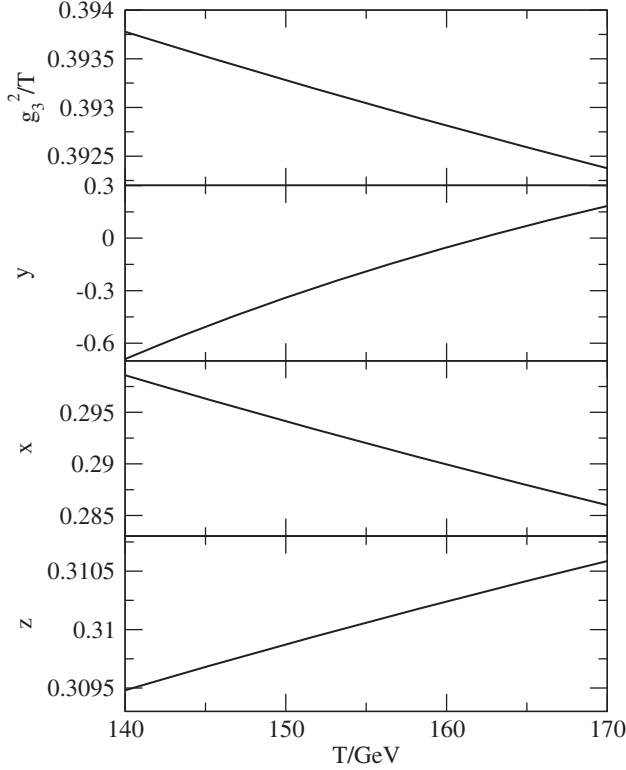


FIG. 1. The parameters of the effective theory (1) as functions of the physical temperature.

III. LATTICE ACTION AND SIMULATIONS

For the lattice implementation it is convenient to introduce a matrix parametrization of the Higgs field by writing

$$\Phi = \frac{1}{g_3^2} \begin{pmatrix} \tilde{\phi}(\phi) \end{pmatrix} \equiv \frac{1}{g_3^2} \begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix}. \quad (5)$$

Under an $SU(2) \times U(1)$ gauge transformation Φ transforms according to

$$\Phi(x) \rightarrow G(x)\Phi e^{-i\theta(x)\sigma_3}. \quad (6)$$

The lattice action corresponding to the continuum theory (1) is

$$\begin{aligned} S = & \beta_G \sum_x \sum_{i < j} \left[1 - \frac{1}{2} \text{Tr} P_{ij} \right] + \frac{\beta_G}{z} \sum_x \sum_{i < j} \frac{1}{2} \alpha_{ij}^2 \\ & - \beta_H \sum_x \sum_i \frac{1}{2} \text{Tr} \Phi^\dagger(x) U_i(x) \Phi(x+i) e^{-i\alpha_i(x)\sigma_3} \\ & + \beta_2 \sum_x \frac{1}{2} \text{Tr} \Phi^\dagger(x) \Phi(x) + \beta_4 \sum_x \left[\frac{1}{2} \text{Tr} \Phi^\dagger(x) \Phi(x) \right]^2. \end{aligned} \quad (7)$$

Here the $SU(2)$ and the (noncompact) $U(1)$ plaquettes are

$$P_{ij}(x) = U_i(x) U_j(x + \hat{i}) U_i^\dagger(x + \hat{j}) U_j^\dagger(x), \quad (8)$$

$$\alpha_{ij}(x) = \alpha_i(x) + \alpha_j(x + \hat{i}) - \alpha_i(x + \hat{j}) - \alpha_j(x). \quad (9)$$

The parameters of the lattice action β_G , β_H , β_2 and β_4 can be expressed in terms of the continuum parameters and lattice spacing a using relations [31,34]

$$\beta_G = \frac{4}{g_3^2 a}, \quad (10)$$

$$\beta_H = \frac{8}{\beta_G}, \quad \beta_4 = \frac{\beta_H^2}{\beta_G}, \quad (11)$$

$$\begin{aligned} \frac{\beta_2}{\beta_H} = & 3 + \frac{8y}{\beta_G^2} - (3 + 12x + z) \frac{\Sigma}{4\pi\beta_G} \\ & - \frac{1}{2\pi^2\beta_G^2} \left[\left(\frac{51}{16} - \frac{9z}{8} - \frac{5z^2}{16} + 9x - 12x^2 + 3xz \right) \right. \\ & \times \left(\log \left(\frac{3\beta_G}{2} \right) + 0.09 \right) \\ & \left. + 5 - 0.9z + 0.01z^2 + 5.2x + 1.7xz \right], \end{aligned} \quad (12)$$

where $\Sigma = 3.17591$. These relations become exact in the limit $a \rightarrow 0$ ($\beta_G \rightarrow \infty$), with an error proportional to a . We note that in 3 dimensions the gauge-Higgs theory has a well-defined continuum limit on the lattice. In Refs. [35,36] partial $O(a)$ improvement of the relations was presented; however, for simplicity we do not implement this here.

Finally, the gauge-invariant lattice observable $\langle \frac{1}{2} \text{Tr} \Phi^\dagger \Phi \rangle$ is related to the renormalized 3-dimensional continuum quantity $\langle \phi^\dagger \phi \rangle$ in the $\overline{\text{MS}}$ scheme by

$$\begin{aligned} \frac{\langle \phi^\dagger \phi \rangle}{g_3^2} = & \left\langle \frac{1}{2} \text{Tr} \Phi^\dagger \Phi \right\rangle - \frac{\Sigma \beta_G}{8\pi} \\ & - \frac{3+z}{16\pi^2} (\log(3\beta_G/2) + 0.6678). \end{aligned} \quad (13)$$

This relation again has corrections at order $O(a)$. The 3-dimensional condensate is related to the physical standard model Higgs condensate v by

$$v^2/T^2 = 2\langle \phi^\dagger \phi \rangle/T. \quad (14)$$

The lattice spacing in physical units is directly fixed by the parameter $\beta_G = 4/(g_3^2 a) \approx 10.2/(aT)$, due to the 3-dimensional nature of the effective theory and the matching procedure used to derive it. Thus, the scale setting, which is often complicated in 4-dimensional lattice field theories, is completely straightforward. We use 3 lattice spacings, $ag_3^2 = 4/\beta_G$, with $\beta_G = 6, 9$ and 16 , corresponding to $aT \approx 1.7, 1.13$ and 0.637 , with volumes

TABLE I. The lattice spacings, volumes and linear sizes used in the analysis. For each lattice, we use 24 temperature values between 140 GeV and 170 GeV. The number of measurements varies between 10 000–40 000, with the largest number of measurements near the cross-over temperature.

$\beta_G = 4/(g_3^2 a)$	volumes/ a^3	$L \times T$
6	24^3	40.7
	48^3	81.4
9	24^3	27.1
	32^3	36.2
	60^3	67.9
16	56^3	35.6
	120^3	76.3

up to $(120a)^3$, as listed in Table I. In physical units the largest lattice sizes are of order 70-80/ T .

The measurements are done at 24 temperature values between 140 and 170 GeV. The simulation algorithm is a mixture of overrelaxation and heat bath update steps, described in detail [without U(1) field] in [37]. The measurements of the observables are done after two repetitions of 4 full overrelaxation update sweeps followed by one heat bath update. The number of measurements varies between 10 000 and 350 000, with most of the measurements done near the cross-over temperature.

IV. HIGGS CONDENSATE AND THE PSEUDOCRITICAL TEMPERATURE

The measurements of the Higgs condensate $\langle \phi^\dagger \phi \rangle$ are extrapolated to the continuum with linear in a extrapolation, as shown in Fig. 2 at a few selected temperatures. For this quantity the finite volume effects are negligible, indistinguishable from the statistical accuracy of the measurements, and we use the largest volumes in our analysis. Below we analyze the finite volume effects of the susceptibility of $\phi^\dagger \phi$ in more detail and show that these also remain very small. Given the presence of the massless U(1) field, the smallness of the finite volume effects is not *a priori* obvious, but is in agreement with the earlier results in [30].

The continuum extrapolation of $\langle \phi^\dagger \phi \rangle$ is shown in Fig. 3. We compare the numerical result with the perturbative broken phase 2-loop Coleman-Weinberg computation [37] and the symmetric phase 3-loop result [26]. In both cases the higher order contributions are estimated with shaded bands. The agreement between the perturbative results and the lattice results is remarkably good, especially in the symmetric phase where the perturbative expansion converges quickly.² There is only a narrow window of a

²Figure 3 can be compared with Fig. 2 in Ref. [26], where the agreement between the lattice and the perturbative results is much weaker, due to the missing continuum limit of the lattice results.

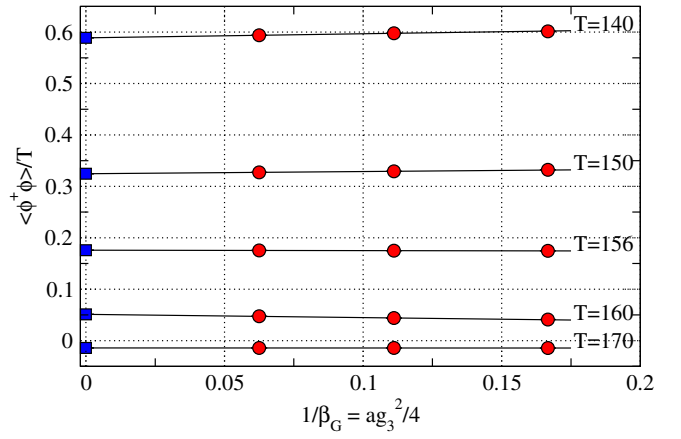


FIG. 2. The continuum limit of $\langle \phi^\dagger \phi \rangle$ at a few selected temperature values. The statistical errors are too small to be visible at this scale.

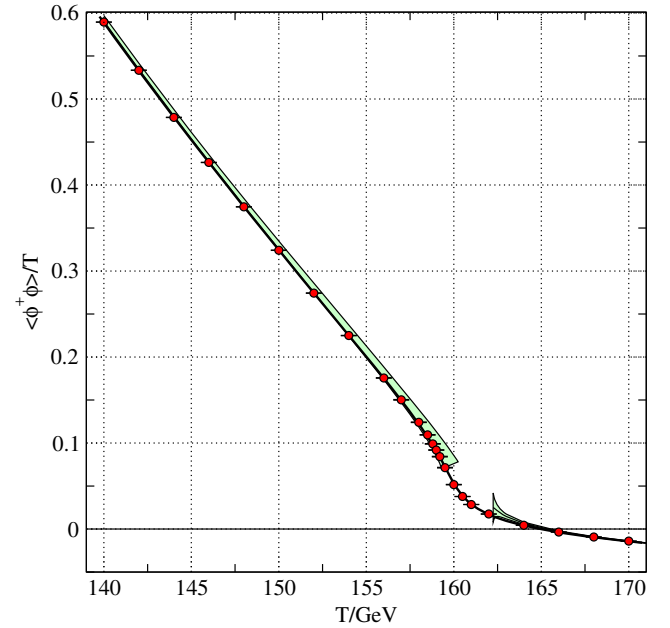


FIG. 3. The continuum result of $\langle \phi^\dagger \phi \rangle$, compared with the perturbative broken and symmetric phase results. The shaded bands are estimations of unknown higher order corrections to perturbative results. The solid continuous line is an interpolation to the data.

few GeV around the cross-over temperature (corresponding to $y \approx 0$) where the perturbative expansions do not converge.

The apparent good convergence in the symmetric phase may be surprising, because in this phase the non-Abelian gauge bosons are perturbatively massless, making the physics at soft momentum scales $k \sim g^2 T$ nonperturbative [8]. The excellent match between the lattice and the perturbation theory means that for the Higgs condensate their effect remains small. This can be contrasted with

e.g. the sphaleron rate, which is in essence completely determined by the soft physics.

We define the pseudocritical temperature by the location of the maximum of the dimensionless susceptibility

$$\chi_{\phi^\dagger\phi} = VT \langle [(\phi^\dagger\phi)_V - \langle (\phi^\dagger\phi)_V \rangle]^2 \rangle, \quad (15)$$

where $(\phi^\dagger\phi)_V = 1/V \int dV \phi^\dagger\phi$ is the volume average of $\phi^\dagger\phi$. This is shown in Fig. 4, for the largest simulation volumes at each lattice spacing. The use of the largest volumes is justified below. There is a well-defined narrow peak near the cross-over temperature, however, the location of the peak has a clear lattice spacing dependence, which can be clearly seen in the zoomed-in subplot of Fig. 4. This makes the conventional way of taking the continuum limit at fixed T difficult: because the peak locations are shifted, at a given fixed T near the peak the measurements of $\chi_{\phi^\dagger\phi}$ at different lattice spacings have a large and nonuniform variation. Indeed, using only linear in a (linear in $1/\beta_G$) continuum extrapolation at each T does not give an acceptable fit, and linear + quadratic in a gives large errors in the extrapolated quantity.

Because the dominant finite a effect is the shift in the peak location, it is plausible that we can obtain a better controlled continuum limit by first “undoing” the shift. In order to illustrate this, let us denote the true continuum limit

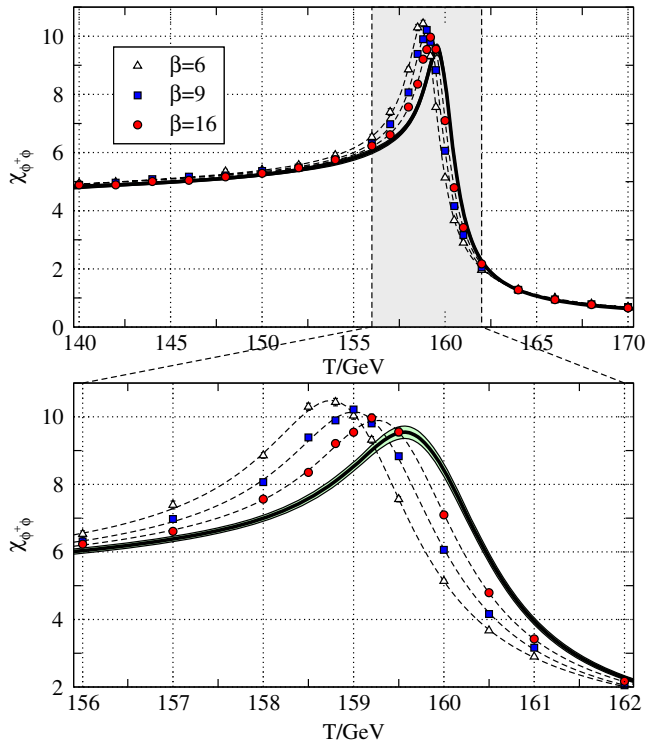


FIG. 4. Above: susceptibility $\chi_{\phi^\dagger\phi}$ shown at $\beta_G = 6, 9$ and 16 , together with the interpolating functions. The continuum limit is shown with a heavy line. Below: As above, zoomed-in to the shaded band near the cross-over region.

by $\chi_{\phi^\dagger\phi}(T)$, and the measured finite- a susceptibility by $\chi_a(T)$. Because the leading cutoff effects are $O(a)$, we can expand χ_a as

$$\chi_a(T) = \chi_{\phi^\dagger\phi}(T) + af(T) + O(a^2). \quad (16)$$

On the other hand, if we perform a shift of the temperature in χ_a by an amount ca , with c a constant, we obtain

$$\begin{aligned} \chi_a(T+ca) &= \chi_a(T) + ca\chi'_a(T) + O(a^2) \\ &= \chi_{\phi^\dagger\phi}(T) + a[c\chi'_a(T) + f(T)] + O(a^2). \end{aligned} \quad (17)$$

Evidently, $\chi_a(T+ca)$ approaches the correct continuum limit $\chi_{\phi^\dagger\phi}(T)$ for any value of c . However, by choosing c suitably it may be possible to reduce the $O(a)$ errors in the extrapolation and hence make the extrapolation better behaved. This is precisely what happens when we choose c so that the peaks of χ_a become aligned.³ A temperature shift of form ca is by no means unique, but it is the simplest choice and it turns out to be sufficient for our purposes.

In order to shift the temperature we need to interpolate the measurements of $\chi_a(T)$ to continuous functions of T . We achieve this by fitting the measurements at each β_G to a differentiable interpolating function, $\chi_{\text{fit}}(T, \beta_G)$. The fit procedure and the choice of the function is described in Appendix A. The fitted functions at each β_G are shown in Fig. 4. We note that commonly used reweighting methods do not work well in this case, because there is limited overlap in the probability distributions between neighboring points over the large temperature range.

We now define the pseudocritical temperature at each β_G by the location of the maxima of the fit functions $\chi_{\text{fit}}(T, \beta_G)$. These are shown in Fig. 5 against $1/\beta_G = ag_3^2/4$. The measurements fall on a remarkably straight line, allowing us to obtain the continuum limit pseudocritical temperature using the fit

$$T_{\chi_{\text{max}}}(\beta_G) = T_c + \frac{C}{\beta_G}. \quad (18)$$

The result is $T_c = 159.58 \pm 0.06$ GeV with $\chi^2 = 0.26$ for the fit. The error bars conservatively include the results from both linear in a and linear + quadratic extrapolation. However, these errors are completely overwhelmed by the per cent-level uncertainties associated with the accuracy of the 3-dimensional effective theory description of the standard model.

³As an aside, an $O(a)$ shift in T (or y in 3-dimensional quantities) is the only unknown quantity in the $O(a)$ improved version of the lattice-continuum relations of the 3-dimensional effective theory [35,36]. The procedure described here can be generalized to determine this improvement coefficient numerically, and $O(a)$ improved lattice-continuum relations would automatically cancel the $O(a)$ shift of the peak location.

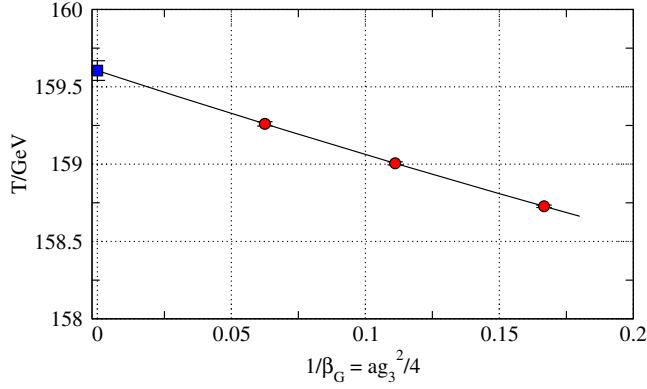


FIG. 5. Continuum extrapolation of the maximum location of $\chi_{\phi^\dagger\phi}$.

We use the above fit to shift the temperature variable of the functions χ_{fit} as described above. Now the peaks of the susceptibility curves shown in Fig. 4 become aligned, and the continuum limit is obtained by making a linear in a extrapolation independently at each shifted T , with average $\chi^2 \approx 1.3$ for 1 degrees of freedom. The final result of the extrapolation is shown in Fig. 4 with a shaded error band, with the errors propagated using the jackknife method.

Finally, in Fig. 6 we show the behavior of the susceptibility at different volumes at $\beta_G = 9$. In this case we have 3 volumes available, with physical linear lattice sizes $L \approx 27/T$, $36/T$ and $68/T$ (at $L/a = 24, 32$ and 60 , respectively). The larger volumes agree within statistical errors, and the smallest volume deviates slightly only at the susceptibility peak. Compatible behavior is seen at other lattice spacings, where the smallest volumes have $L \gtrsim 36/T$. Thus, we conclude that the largest volumes have negligible finite volume corrections for the Higgs condensate and its susceptibility.

In the absence of massless modes the finite volume effects vanish exponentially in L . In this case the system has a massless $U(1)$ gauge boson, and thus one could expect to observe power law finite volume effects. However, the hypercharge $U(1)$ field participates weakly to the transition, only by mixing with W^3 by the Weinberg angle. In part, this explains the smallness of the finite volume effects.⁴ Similar behavior has been observed in earlier lattice studies [30].

In order to facilitate the detailed analysis of thermodynamic functions we also interpolate the $\phi^\dagger\phi$ -measurements. We do this by interpolating at each lattice spacing using a spline fit, shifting the temperatures by the same amount as for $\chi_{\phi^\dagger\phi}$ and extrapolate to continuum. The result is included in Fig. 3 as a continuous line. The temperature shift before the continuum extrapolation changes the result by a very small amount which is not distinguishable in the

⁴Strong external hypermagnetic fields do nevertheless affect the transition, see e.g. [31].

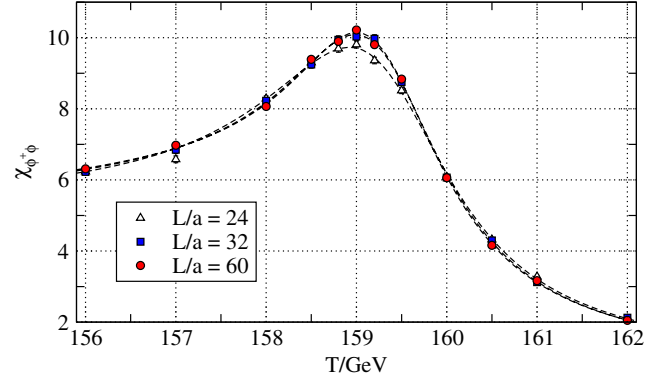


FIG. 6. The finite volume behavior of the susceptibility $\chi_{\phi^\dagger\phi}$, shown for the lattices of size $L/a = 24, 32$ and 60 at $\beta_G = 9$. Only the smallest volume shows appreciable deviation, which is much below the lattice spacing effects in Fig. 4.

figure. For possible future uses the interpolated results for $\phi^\dagger\phi$ and $\chi_{\phi^\dagger\phi}$, together with other thermodynamic measurements, can be downloaded from [33].

V. THERMODYNAMICS OF THE CROSS-OVER

Recently, Laine and Meyer [26] showed how one can combine perturbative calculations and effective theory simulation to obtain the standard model pressure, energy density and other thermodynamic quantities derived from these. As an input they used the simulation results from Ref. [20]. Because these results use only a single lattice spacing and ignore the hypercharge $U(1)$, we revisit the calculation using our improved data.

The fundamental thermodynamic quantity is the interaction measure (“trace anomaly”)

$$\Delta \equiv \frac{e(T) - 3p(T)}{T^4} = T \frac{d}{dT} \frac{p(T)}{T^4}. \quad (19)$$

As described in Appendix B, the interaction measure can be expressed in the following form:

$$\Delta(T, \bar{\mu}) = A(T; \bar{\mu}) + B(T; \bar{\mu}) \frac{\langle \phi^\dagger\phi \rangle(T)}{T}, \quad (20)$$

where $\bar{\mu}$ is the $\overline{\text{MS}}$ renormalization scale, and the functions A and B are given in Appendix B. The function A has been calculated up to parametric order g^5 [26], however, we use the $O(g^4)$ expression, because it has been argued that the $O(g^5)$ contribution, which is mostly due to QCD contributions, leads to an underestimate of the pressure and the energy density. The resulting Δ is shown in Fig. 7 with error band obtained through jackknife analysis.

From Fig. 3 we can see that the direct perturbative computation of $\langle \phi^\dagger\phi \rangle(T)$ fails to converge near the cross-over temperature. On the other hand, the functions $A(T; \bar{\mu})$

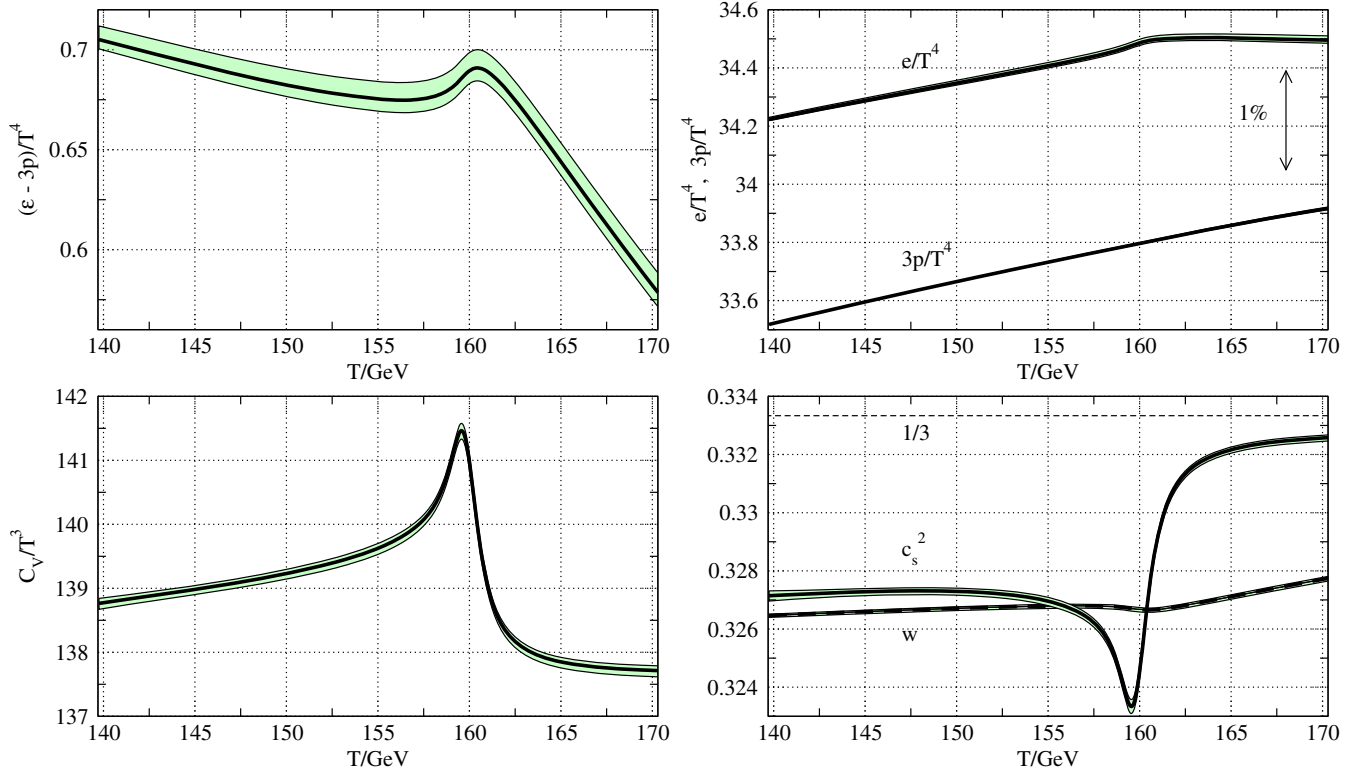


FIG. 7. The interaction measure $\Delta = (e - 3p)/T^4$ (top left); energy density e and pressure p (top right); heat capacity $C_V = de/dT$ (bottom left); speed of sound squared c_s^2 and the equation of state parameter $w = p/e$ (bottom right). The error bands are a combination of the statistical errors and renormalization scale variation $\bar{\mu} = (0.5 \dots 2)\pi T$. The energy density and the pressure are affected by a systematic uncertainty of order 1%, indicated with a vertical arrow.

and $B(T; \bar{\mu})$ do not suffer from this problem. Therefore, by measuring the Higgs condensate nonperturbatively on the lattice, we obtain well-defined expressions for thermodynamic quantities across the cross-over.

The pressure is obtained from Δ by integration:

$$\frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_{T_0}^T dT' \frac{\Delta(T'; \bar{\mu})}{T'}. \quad (21)$$

In order to evaluate this we need to fix the pressure at a reference temperature T_0 . We use the results in Refs. [26,38] at the lowest temperature in our temperature range, $T_0 = 140$ GeV: $p(T_0)/T_0^4 = 11.173$. The estimated uncertainty in this value is of order 1%; we discuss this together with other systematic errors at the end of this section.

From $\Delta(T)$ and $p(T)$ we can obtain other thermodynamic functions: the energy density $e/T^4 = \Delta + 3p/T^4$, the entropy density $s = p' = (e + p)/T$, the heat capacity $C_V/T^3 = e'/T^3 = 7\Delta + T\Delta' + 12p/T^4$, the speed of the sound squared $c_s^2 = p'/e'$ and the equation of the state parameter $w = p/e$. The quantities involving Δ' can in principle be calculated directly from (20) by numerical differentiation. However, we obtain a numerically more robust result by realizing that to the parametric accuracy needed

$$\frac{d}{dT} \frac{\langle \phi^\dagger \phi \rangle}{T} \approx \frac{dy}{dT} \frac{d}{dy} \frac{\langle \phi^\dagger \phi \rangle}{T} = -\frac{dy}{dT} \chi_{\phi^\dagger \phi}. \quad (22)$$

The direct numerical differentiation of $\langle \phi^\dagger \phi \rangle$ gives comparable results, but because it was obtained through continuum extrapolation of spline interpolations the derivative of it becomes somewhat jagged.

The final results are collected in Fig. 7, with error bands which combine quadratically the statistical errors and the variation in the renormalization scale $\bar{\mu} = (0.5T \dots 2)\pi T$. The renormalization scale dominates the error bands except in quantities where Δ' contributes, where the statistical errors are of comparable magnitude. The results show similar features than the earlier ones published in Ref. [26], but with improved accuracy and reliability, because the Higgs condensate and susceptibility have been properly extrapolated to the continuum. The tabulated numerical results can be downloaded from [33].

Figure 7 clearly shows that the cross-over region, defined as the temperature range where the thermodynamic variables deviate from their (perturbative) hot or cold phase behavior, is narrow, between ~ 157 and 162 GeV. This is precisely the region where perturbation theory does not converge and lattice simulations are needed. The variation of the thermodynamic quantities across the cross-over is

very mild, only at few per cent level. This is a natural consequence of the fact that only a couple of degrees of freedom of the standard model are strongly sensitive to the cross-over. The behavior is markedly different from the QCD cross-over, see e.g. [39] for a recent review.

In addition to the statistical errors and renormalization scale ambiguity shown in Fig. 7, there are theoretical uncertainties that can be estimated to be at percent level [26], especially affecting the energy density and the pressure. As an indication of these a 1% bar is shown in Fig. 7, and it clearly dominates over the error bands of e and p . This uncertainty mostly pertains to absolute normalization in these quantities, whereas the shape of the curves can be expected to be preserved to much better accuracy. In particular, the characteristic features of the thermodynamic functions around the cross-over are mostly due to the evolution of $\langle\phi^\dagger\phi\rangle$ and hence more accurately determined.

VI. SCREENING MASSES

The screening masses of the Higgs and SU(2) gauge fields are direct probes of the “softening” of the physics at the cross-over. We measure the Higgs, W^3 and hypercharge B spatial correlation functions and extract the screening masses (inverse correlation lengths) as functions of the temperature. The Higgs and W^3 correlation functions are measured with lattice operators corresponding to $H = \text{Tr}\Phi^\dagger\Phi$ and $W_i^3 = \text{Tr}\Phi^\dagger iD_i\Phi\sigma_3$, projected to zero momentum. For example, the Higgs correlation function is (here in units of lattice spacing a):

$$C_H(z) = \frac{1}{N^3} \sum_{x_i, y_i, z'} \langle H(x_1, y_1, z') H(x_2, y_2, z' + z) \rangle, \quad (23)$$

where x_i, y_i, z, z' are (x, y, z) coordinates compatible with the periodic boundary conditions. The local operators corresponding to W^+ and W^- (linear combinations of $\text{Tr}\Phi^\dagger iD_i\Phi\sigma_a$ with $a = 1, 2$) are not gauge invariant under U(1) and we do not measure them here. To enhance the signal, we use lattice operators which are smeared and blocked recursively 3 times, as described in Ref. [30]. The screening masses extracted from the correlation functions are shown in Fig. 8. We use the largest lattices at $\beta_G = 9$ and $\beta_G = 16$, with results which are consistent within the statistical errors.

In the broken phase, where $\phi^\dagger\phi$ has a large expectation value, the operators couple appropriately to the physical Higgs and Z-boson excitations. At small temperatures the Higgs screening mass is larger than that of W^3 , whereas near the cross-over temperature the Higgs becomes sensitive to the near-critical behavior and the mass becomes small but still remains nonzero. Even at its largest, the Higgs correlation length is smaller than $10/T$, which is

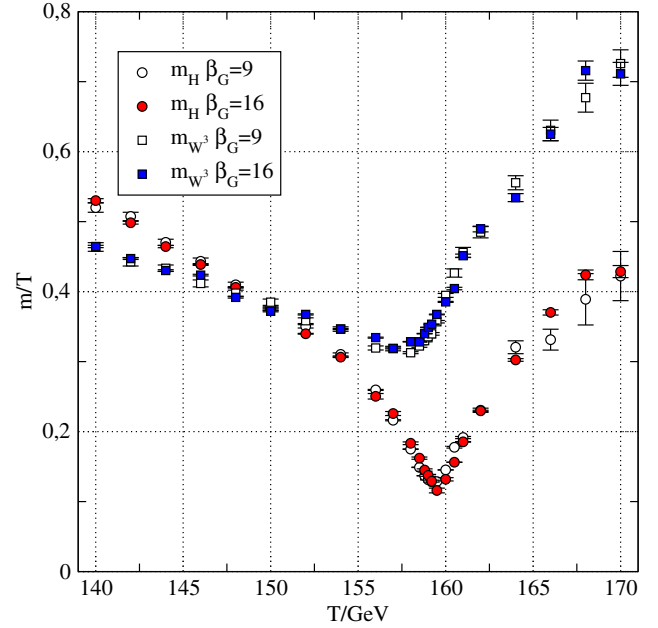


FIG. 8. Higgs and W^3 screening masses, measured from 60^3 , $\beta_G = 9$ and 120^3 , $\beta_G = 16$ lattices.

substantially smaller than the largest lattice sizes $\sim 70\text{--}80/T$.

In the symmetric phase, the non-Abelian gauge fields are confining, and the operators couple to bound states of two scalars. The correlation functions become noisy and the screening masses increase rapidly.

The U(1) gauge field correlation function can be used to measure the γ -Z mixing, i.e. the effective Weinberg angle. We define the operator

$$O_{\mathbf{p}}(z) = \sum_{x_1, x_2} \alpha_{12}(x_1, x_2, z) e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (24)$$

where the sum is taken over the plane (x_1, x_2) , α_{ij} is the (noncompact) hypercharge U(1) plaquette (9) and \mathbf{p} is a transverse momentum vector compatible with periodic boundary conditions: $(p_1, p_2, p_3) = 2\pi/N(n_1, n_2, 0)$ with integer n_1 and n_2 . In our measurements we use the smallest nonvanishing momentum, with $|\mathbf{p}| = 2\pi/N$. At $\mathbf{p} = 0$ the operator $O_{\mathbf{p}}$ vanishes, due to the periodic boundary conditions. The correlation function

$$G(z) = \frac{1}{N^3} \sum_t \langle O_{\mathbf{p}}(t) O_{\mathbf{p}}^*(z+t) \rangle \quad (25)$$

has the long distance behavior [30]

$$G(z) \rightarrow \frac{A_\gamma z}{2\beta_G} \frac{ap^2}{\sqrt{p^2 + m_\gamma^2}} e^{-z\sqrt{p^2 + m_\gamma^2}} \quad (26)$$

where m_γ is the photon screening mass and A_γ gives the projection of the operator to the hypercharge U(1) field, in

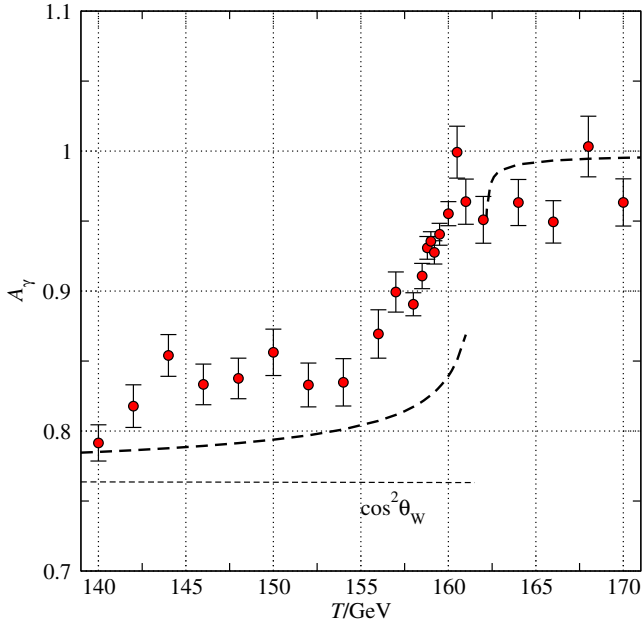


FIG. 9. The effective γ - Z mixing as a function of the temperature. The dashed lines show the 1-loop perturbative results.

effect yielding the temperature-dependent effective mixing angle. At tree level, $A_\gamma = 1$ in the symmetric phase and $A_\gamma = \cos^2 \theta_W$ in the broken phase.

The photon screening mass m_γ vanishes within our measurement accuracy at all temperatures. The projection A_γ is shown in Fig. 9 for $\beta_G = 9$, 60^3 lattice. The measurement is noisy, but we can observe that $A_\gamma \approx 1$ in the symmetric phase down to the cross-over temperature, and it starts to decrease as the Higgs field expectation value grows at lower temperatures, slowly approaching the tree-level value.

Beyond tree-level perturbative estimates for the behavior of A_γ can be obtained by calculating at 1-loop order the residue of the $1/k^2$ pole in the $\langle B_i B_j \rangle$ correlator. In the symmetric and broken phases one obtains [30]

$$A_\gamma^{\text{symm}} = 1 - \frac{z}{48\pi\sqrt{y}} \quad (27)$$

$$A_\gamma^{\text{broken}} = \cos^2 \theta_W \left(1 + \frac{11}{12} \frac{g_3^2 \sin^2 \theta_W}{\pi m_W} \right) \quad (28)$$

where m_W is the perturbative W mass. These expressions clearly anticipate the behavior we observe on the lattice, although they diverge as $y \rightarrow 0 \pm$.

VII. CONCLUSIONS

We have accurately determined the Higgs field expectation value and its susceptibility across the standard model cross-over using lattice simulations of an effective

3-dimensional theory. Defining the cross-over temperature by the maximum of the susceptibility, we obtain $T_c = 159.6 \pm 0.1 \pm 1.5$ GeV, where the first error is due to the statistical accuracy of the lattice computation and the second one is the estimated uncertainty of the effective theory approach [16,26]. Following the approach of Laine and Meyer [26], these results were used to obtain the behavior of basic thermodynamic quantities, including energy density, pressure, heat capacity and the speed of sound, across the cross-over. There is a well-defined cross-over region where thermodynamic quantities deviate from the low- or high-temperature behavior. This region is quite narrow, between 157 and 162 GeV. The results are consistent with the standard picture of the electroweak cross-over: Higgs and W modes become softer but not critical, and the $U(1)$ field remains massless at all temperatures.

Overall our results are compatible with the analysis in Ref. [26] using lattice data from Ref. [20]. However, our results are significantly improved numerically: we have much larger volumes with higher statistical accuracy, the data is extrapolated to the continuum and we include the $U(1)$ field in the effective theory. Thus, our results form an important consistency and reliability check of the earlier results.

For phenomenological applications the thermodynamic quantities here can be combined with existing low- [38] and high-temperature [27] perturbative results. This has been done in Ref. [26] and we do not repeat this analysis here.

The errors are dominated by systematic uncertainties at per cent level, but since these are expected to be roughly constant at the temperature range considered, they would mostly shift the scale of the results without changing their behavior. The cross-over is clearly visible in distinct features in the heat capacity, the speed of sound and the interaction measure. The relative magnitude of these features is small, only at few % level, which already follows from the fact that only few degrees of freedom of the standard model are sensitive to the early stages of the transition.

In Ref. [20] the central value of the cross-over temperature was reported to be 159.5 GeV, in practice equal to our continuum result here. This work was done without the $U(1)$ field and only at one lattice spacing, $\beta_G = 9$, using relatively small statistics. However, one cannot make the conclusion that the effects of the $U(1)$ field can be neglected in the transition: from Fig. 6 we see that at $\beta_G = 9$ the maximum of the susceptibility is at ≈ 159.0 GeV, which is a numerically significant deviation from the continuum result. Thus, the apparent excellent agreement is partly coincidental, due to a different definition of the cross-over temperature and limited accuracy in [20]. Nevertheless, we can estimate that the presence of the $U(1)$ field in the effective theory shifts the cross-over temperature $\lesssim 0.5$ GeV.

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APPENDIX A: INTERPOLATING FUNCTION FOR THE SUSCEPTIBILITY

In this subsection we describe the method used for the interpolation of the susceptibility measurement at different lattice volumes and β_G -values in Sec. IV. Instead of the temperature, it is convenient to do the interpolation in terms of the 3-dimensional variable $y = m_3^2/g_3^2$, which is actually the variable used internally throughout our analysis.

The functional form of the fit is motivated by the sharp susceptibility peak near $y = 0$ ($T = 162.1$ GeV), approaching almost flat behavior away from the peak. Thus, we construct a differentiable function which approaches power series in $1/y$ on both sides of the cross-over. For our data we obtain very good results with the function

$$\chi_{\text{fit}}(y, \beta_G) = \frac{(\sum_{i=0}^{n_1} c_i s^i) E_1(y) + (\sum_{i=0}^{n_2} d_i s^i) E_2(y)}{E_1(y) + E_2(y)}, \quad (\text{A1})$$

where

$$\begin{aligned} y_s &= y - a_1 \\ s &= 1/\sqrt{y_s^2 + a_2} \\ E_1(y) &= 1/E_2(y) = e^{-a_3 y_s}. \end{aligned}$$

Here $a_1 \dots a_3$, $c_0 \dots c_{n_1}$ and $d_0 \dots d_{n_2}$ are fit parameters, s corresponds to a regulated $|y|^{-1}$ and the exponential factors $E_{1,2}$ are introduced to smoothly switch between the c -series on the low temperature side and the d -series on the high temperature side of the cross-over. Parameter a_1 corresponds roughly to the location of the peak, and $a_2 > 0$ regulates the singular behavior at $y_s \rightarrow 0$.

Using $n_1 = 3$, $n_2 = 2$ we obtain a good fit to all measurements ($\chi^2/\text{d.o.f.} = 13/16$, $20/16$ and $10/16$ for the largest volumes at $\beta_G = 6$, 9 and 16 , respectively), as shown in Fig. 4. Adding higher order terms does not

improve the fit, and the final function remains practically identical, with variation remaining well within the statistical errors.

The fit function is not unique, and the fit parameters have no direct physical meaning. The function only serves to interpolate the measurements and cannot be extended beyond the range of data shown here. Naturally, significantly more accurate data or extended temperature range would require a more general fit function or some other means of interpolation. The error propagation is done with jackknife analysis.

APPENDIX B: MATCHING OF THE INTERACTION MEASURE

For completeness, we summarize here the computation of the interaction measure and the structure of the coefficients $A(T, \bar{\mu})$ and $B(T; \bar{\mu})$ in Eq. (20). These results have been calculated in Ref. [26], which we follow and refer to for all details.

The only dimensionful parameter in the standard model Lagrangian is the Higgs mass parameter ν . Thus, for dimensional reasons, the physical pressure of the standard model in the $\overline{\text{MS}}$ scheme can be written as

$$\frac{p(T)}{T^4} = \hat{p}_R \left(\frac{\bar{\mu}}{T}, \frac{\nu^2(\bar{\mu})}{T^2}, g^2(\bar{\mu}) \right) - \frac{p_{0R}(\bar{\mu}, \nu^2(\bar{\mu}), g^2(\bar{\mu}))}{T^4}. \quad (\text{B1})$$

Here $\bar{\mu}$ is the $\overline{\text{MS}}$ scale, $\hat{p}_R = p_R/T^4$, and p_{0R} is the zero-temperature pressure, which subtracts UV divergences. Now the interaction measure Δ is

$$\begin{aligned} \Delta &= T \frac{d}{dT} \frac{p(T)}{T^4} \\ &= -\frac{\partial \hat{p}_R}{\partial \ln(\bar{\mu}/T)} - \frac{2\nu^2 [Z_m \langle \phi^\dagger \phi \rangle_{4d}]_R}{T^4} + \frac{4p_{0R}}{T^4} \\ &\equiv \Delta_1 + \Delta_2 + \Delta_3 \end{aligned} \quad (\text{B2})$$

Thus, the computation of the interaction measure is split into three parts: using the terminology of Ref. [26], these are breaking of the scale invariance by quantum corrections, temperature evolution of the Higgs condensate (explicit breaking of the scale invariance) and the vacuum subtraction term. Here $\langle \phi^\dagger \phi \rangle_{4d}$ is the expectation value of the 4-dimensional standard model Higgs condensate.

The first contribution to Δ can be written as

$$\begin{aligned} \Delta_1(T) &= \frac{1}{(4\pi)^2} \left[\frac{198 + 141n_G - 20n_G^2}{54} g_s^4 + \frac{266 + 163n_G - 40n_G^2}{288} g_W^4 - \frac{144 + 375n_G + 1000n_G^2}{7776} g_1^4 - \frac{g_W^2 g_1^2}{32} \right. \\ &\quad \left. - h_t^2 \left(\frac{7h_t^2}{32} - \frac{5g_s^2}{6} - \frac{15g_W^2}{64} - \frac{85g_1^2}{576} \right) - \lambda \left(\lambda + \frac{h_t^2}{2} - \frac{g_1^2 + 3g_W^2}{8} \right) + \frac{\nu^2}{T^2} \left(h_t^2 + 2\lambda - \frac{g_1^2 + 3g_W^2}{4} \right) - \frac{2\nu^4}{T^4} \right], \end{aligned} \quad (\text{B3})$$

where $n_G = 3$ is the number of generations, h_t is the top-Yukawa coupling and g_1 , g_W and g_S are the U(1), SU(2) and SU(3) gauge couplings and λ the Higgs self-coupling of the standard model. Defining the leading order Debye mass parameters

$$m_{E1} = \sqrt{\left(\frac{1}{6} + \frac{5n_G}{9}\right)g_1^2 T^2}, \quad m_{E2} = \sqrt{\left(\frac{5}{6} + \frac{n_G}{3}\right)g_W^2 T^2}, \quad (B4)$$

Δ_2 becomes

$$\begin{aligned} \Delta_2(T) = & -\frac{2\nu^2}{T^2} \left\{ 1 + \frac{3}{2(4\pi)^2} \left[(g_1^2 + 3g_W^2 - 8\lambda) \ln \frac{\bar{\mu} e^{\gamma_E}}{4\pi T} - 4h_t^2 \ln \frac{\bar{\mu} e^{\gamma_E}}{\pi T} \right] \right\} \frac{\langle \phi^\dagger \phi \rangle}{T} \\ & - \frac{\nu^2}{3T^2} \left\{ 1 - \frac{3}{2(4\pi)^2} \left[(g_1^2 + 3g_W^2) \left(4 \ln \frac{g_3^2}{\bar{\mu}} + 3 \ln \frac{\bar{\mu}}{4\pi T} + \gamma_E + \frac{5}{3} + \frac{2\zeta'(-1)}{\zeta(-1)} \right) + 4h_t^2 \ln \frac{\bar{\mu} e^{\gamma_E}}{8\pi T} + 8\lambda \ln \frac{\bar{\mu} e^{\gamma_E}}{4\pi T} \right] \right\} \\ & + \frac{2\nu^2}{(4\pi)^3 T^2} \left[\frac{g_1^2 m_{E1} + 3g_W^2 m_{E2}}{T} + \frac{g_1^4 T}{16m_{E1}} + \frac{3g_1^2 g_W^2 T}{4(m_{E1} + m_{E2})} + \frac{g_W^4 T}{2m_{E2}} \left(\frac{35}{24} + \ln \frac{2m_{E2}}{g_3^2} \right) \right]. \end{aligned} \quad (B5)$$

Here $\langle \phi^\dagger \phi \rangle$ is the 3-dimensional Higgs condensate, which contains nonperturbative soft physics and is evaluated on the lattice.

The parameters appearing in the expressions of Δ_1 and Δ_2 are fixed at scale $\bar{\mu} = m_W$ and evolved to scale $\bar{\mu}$, which is varied between $0.5 \dots 2\pi T$, using 1-loop RG relations to be found e.g. in Ref. [16]. Finally, the vacuum contribution $\Delta_3(T, \bar{\mu}) = 4p_{\text{OR}}/T^4$ can be solved directly with the help of the simultaneous RG evolution [26]:

$$\bar{\mu} \frac{dp_{\text{OR}}}{d\bar{\mu}} = \frac{\nu^4}{8\pi^2}. \quad (B6)$$

The three contributions to $\Delta(T)$ can now be arranged to functions A and B in Eq. (20) as follows:

$$A(\bar{\mu}; T) = \Delta_1 + \Delta_3 + \Delta_2^A \quad (B7)$$

$$B(\bar{\mu}; T) = \Delta_2^B, \quad (B8)$$

where Δ_2^B is the expression multiplying $\langle \phi^\dagger \phi \rangle/T$ in Eq. (B5) and Δ_2^A includes the rest of Eq. (B5).

For Δ_2 and Δ_3 , the leading corrections to Eqs. (B5) and (B6) are of parametric order g^6 ($\sim h_t^6 \sim \lambda^3$). Δ_1 is known up to parametric order $O(g^5)$; however, we omit the $O(g^5)$ contribution, which is mostly due to QCD interactions, because it has been argued that their inclusion leads to numerical underestimation of the pressure and the energy density [26].

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